Ascertaining the Values of σ_x, σ_y, and σ_z of a Polarization Qubit

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In the 1987 spin-retrodiction puzzle of Vaidman, Aharonov, and Albert one is challenged to ascertain the values of σ_x, σ_y, and σ_z of a spin-$\frac{1}{2}$ particle by utilizing entanglement. We report the experimental realization of a quantum-optical version in which the outcome of an intermediate polarization projection is inferred by exploiting single-photon two-qubit quantum gates. The experimental success probability is consistently above the 90.2% threshold of the optimal one-qubit strategy, with an average success probability of 95.6%.

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The 1987 paper [1] by Vaidman, Aharonov, and Albert (VAA) answered the question of "how to ascertain the values of σ_x, σ_y, and σ_z of a spin-$\frac{1}{2}$ particle" and so showed, in the words of Mermin, how to perform the following trick: Alice prepares a quantum mechanical system in a certain initial state and gives it to Bob. Without telling Alice his choice, Bob measures σ_x, σ_y, or σ_z of a spin-$\frac{1}{2}$ particle contained in the system, and gives the system back to Alice, who makes an additional measurement. This enables her (still not knowing Bob's choice) to announce correctly what Bob's result was if he measured σ_x, what it was if he measured σ_y, and what it was if he measured σ_z [2].

Thanks to Aharonov's popularization, this spin-retrodiction challenge became generally known as the *mean king's problem*. It embeds the VAA puzzle into a colorful tale where Alice is a shipwrecked physicist and Bob the underling of the physicist-hating despot who rules the remote island on which Alice got stranded [3].

For the quantum-optical realization we employ the polarization of single photons as the VAA puzzle's degree of freedom of spin-$\frac{1}{2}$ type. The standard for judging the experimental results is set by the optimal strategy that Alice can follow if she manipulates solely the single polarization qubit in question. Photon states with horizontal and vertical polarization (h or v) are identified with the eigenstates of σ_z, right and left circular polarization states (r or l) with those of σ_y, and linear polarization states under $+45°$ and $-45°$ ($+$ or $-$) with those of σ_x,

$$\sigma_x = |+\rangle\langle+| - |-\rangle\langle-|, \qquad \sigma_y = |\mathsf{r}\rangle\langle\mathsf{r}| - |\mathsf{l}\rangle\langle\mathsf{l}|, \qquad \sigma_z = |\mathsf{h}\rangle\langle\mathsf{h}| - |\mathsf{v}\rangle\langle\mathsf{v}|. \tag{1}$$

Imagine now that Alice just prepares the photon in a certain polarization state—right circular polarization (r), say. She can then surely infer the correct answer if Bob measures σ_y. At the final stage, she measures $\sigma_x + \sigma_z$, thereby finding the photon linearly polarized either halfway between h and $+$ or halfway between v and $-$. Although she lacks perfect retrodiction if Bob measured σ_x or σ_z, she can guess his measurement result rather well, namely, with total betting odds of $\frac{1}{3}(2 + 2^{-1/2}) = 90.2\%$. In fact, one demonstrates easily [4] that this is the largest likelihood for guessing right that she can achieve by such a single-qubit strategy.

To do better than these 90.2%, Alice takes to heart the advice given by the VAA trio and entangles the photon polarization with an auxiliary qubit (which is not revealed to Bob). Alice's final projection of the auxiliary qubit and the photon returned by Bob onto an entangled basis allows perfect polarization retrodiction and so enables her to solve the mean king's problem.

An essential ingredient of all variants of the VAA puzzle and its various generalization [2,3,5–7] is that the intermediate measurement by Bob is an ideal von Neumann measurement that finds an eigenvalue of the observable in question and leaves the system behind in the respective eigenstate. In the present optical experiment, we use projections as equivalent replacements of von Neumann measurements [8]. For the photons that are successfully projected by Bob, Alice faces the original VAA problem of determining which projection occurred at the intermediate stage.

At the first stage of the experiment, Alice prepares the entangled two-qubit state. In our experiment (see Fig. 1), the auxiliary qubit is a longitudinal spatial mode of the photon, namely, the binary alternative of being early or late (E or L) which, entangled with the polarization of the photon, is in the single-photon two-qubit state

$$|\mathsf{init}\rangle = 2^{-1/2}(|\mathsf{E},\mathsf{h}\rangle + |\mathsf{L},\mathsf{v}\rangle), \tag{2}$$

where $|\mathsf{E},\mathsf{h}\rangle$, for instance, denotes a horizontally polarized photon that arrives early. This additional E/L qubit is hidden from Bob who does not know the precise instant when the photon is ready.

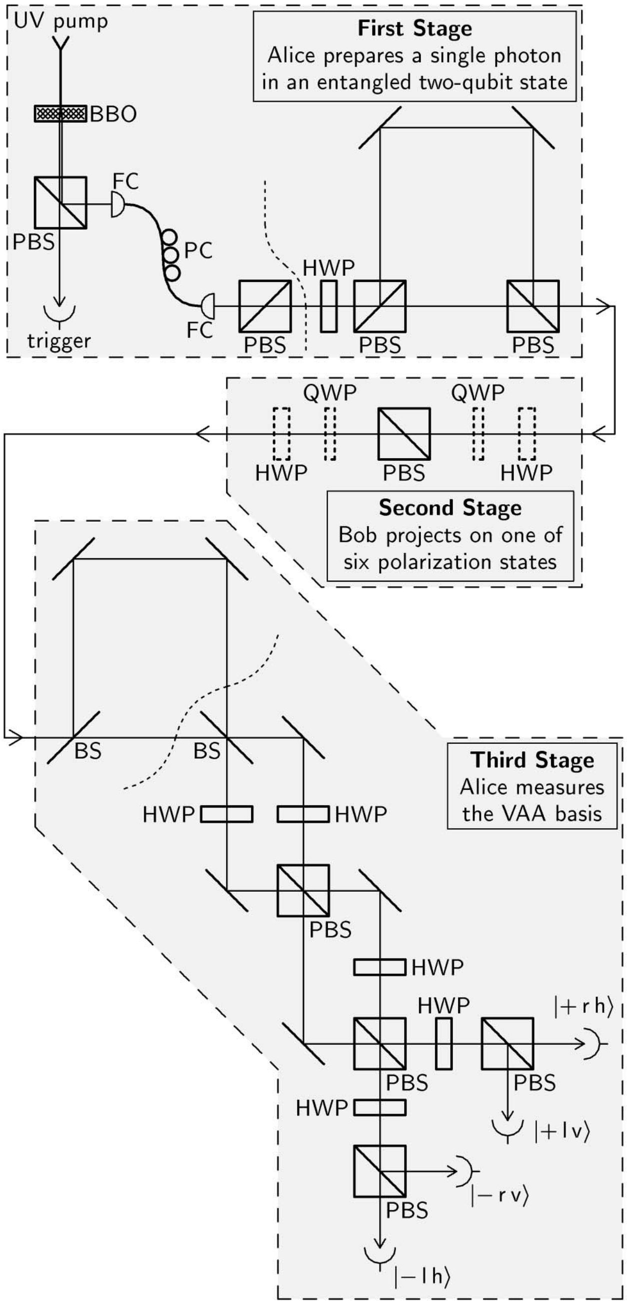

FIG. 1. The three stages of the quantum-optical experiment that realizes the mean king's problem.

At the second stage, Bob projects the entangled state of (2) onto one of six product states, depending on the polarization he actually selects,

$$\sigma_x{:}|\mathsf{init}\rangle \rightarrow \begin{cases} |'{+}'\rangle \equiv 2^{-1/2}(|\mathsf{E},+\rangle + |\mathsf{L},+\rangle), \\ |'{-}'\rangle \equiv 2^{-1/2}(|\mathsf{E},-\rangle - |\mathsf{L},-\rangle), \end{cases}$$
$$\sigma_y{:}|\mathsf{init}\rangle \rightarrow \begin{cases} |'\mathsf{r}'\rangle \equiv 2^{-1/2}(|\mathsf{E},\mathsf{r}\rangle - i|\mathsf{L},\mathsf{r}\rangle), \\ |'\mathsf{l}'\rangle \equiv 2^{-1/2}(|\mathsf{E},\mathsf{l}\rangle + i|\mathsf{L},\mathsf{l}\rangle), \end{cases} \tag{3}$$
$$\sigma_z{:}|\mathsf{init}\rangle \rightarrow \begin{cases} |'\mathsf{h}'\rangle \equiv |\mathsf{E},\mathsf{h}\rangle, \\ |'\mathsf{v}'\rangle \equiv |\mathsf{L},\mathsf{v}\rangle. \end{cases}$$

Then, at the third stage, Alice performs a measurement in which she distinguishes the four mutually orthogonal states of the VAA basis that are given by

$$\begin{aligned} |+\mathsf{rh}\rangle &= 2^{-1/2}|\mathsf{E},\mathsf{h}\rangle + \tfrac{1}{2}i^{1/2}|\mathsf{E},\mathsf{v}\rangle + \tfrac{1}{2}i^{-1/2}|\mathsf{L},\mathsf{h}\rangle, \\ |+\mathsf{lv}\rangle &= 2^{-1/2}|\mathsf{L},\mathsf{v}\rangle + \tfrac{1}{2}i^{-1/2}|\mathsf{E},\mathsf{v}\rangle + \tfrac{1}{2}i^{1/2}|\mathsf{L},\mathsf{h}\rangle, \\ |-\mathsf{rv}\rangle &= 2^{-1/2}|\mathsf{L},\mathsf{v}\rangle - \tfrac{1}{2}i^{-1/2}|\mathsf{E},\mathsf{v}\rangle - \tfrac{1}{2}i^{1/2}|\mathsf{L},\mathsf{h}\rangle, \\ |-\mathsf{lh}\rangle &= 2^{-1/2}|\mathsf{E},\mathsf{h}\rangle - \tfrac{1}{2}i^{1/2}|\mathsf{E},\mathsf{v}\rangle - \tfrac{1}{2}i^{-1/2}|\mathsf{L},\mathsf{h}\rangle, \end{aligned} \tag{4}$$

where $i^{\pm 1/2} = (1 \pm i)/\sqrt{2}$. Upon detecting the second VAA state $|+\mathsf{lv}\rangle$, for example, Alice would infer that Bob projected on the + polarization if the choice was between the σ_x alternatives of + and −, that he projected on l if the choice was between r and l, and that he projected on v if it was the σ_z choice between h and v. Her inference is always correct because $|+\mathsf{lv}\rangle$ is orthogonal to $|'{-}'\rangle$, $|'\mathsf{r}'\rangle$, and $|'\mathsf{h}'\rangle$ by construction.

The setup of the first stage of the experiment, in which Alice prepares the two-qubit state (2), is sketched at the top of Fig. 1. To generate a single photon with a well defined emission time, she first produces a pair of simultaneously emitted photons, then detects one of them to record the time of emission, and uses the other for the polarization-retrodiction experiment. To the left of the dashed line, we have first the nonlinear BBO crystal, in which an incoming ultraviolet (UV) photon is converted into a pair of copropagating infrared photons—one h polarized, the other v polarized (parametric down conversion of type II). This pair is split at a polarizing beam splitter (PBS), which transmits the h photon and reflects the v photon. The h photon is detected, and this gives us the trigger signal by which the detectors of the third stage (bottom part of Fig. 1) are gated. To ensure single-mode operation, the v photon is fed into a single-mode fiber through a fiber coupler (FC). Upon emerging from the fiber through another FC the photon passes through a PBS that selects h polarization. The fiber is equipped with a polarization control (PC) to manipulate the photon polarization such that the yield of this selection is maximized.

Accordingly, a photon that makes it to the dashed line at the top of Fig. 1 is assuredly h polarized. It passes through a half-wave plate (HWP) that changes the polarization state to $|+\rangle = 2^{-1/2}(|\mathsf{h}\rangle + |\mathsf{v}\rangle)$. The photon then traverses an unbalanced Mach-Zehnder interferometer (MZI) that has PBSs at the entry and exit ports. As a consequence, the h component takes the short way and emerges early (E), and the v component takes the long way and is late (L). The photon amplitude is thereby split longitudinally, because the detour of about 90 cm is longer than the coherence length of the photon (~ 0.1 mm). The photon is now prepared in the single-photon two-qubit state (2), and Alice hands it over to Bob. Without knowing the trigger time for reference, it is impossible for him to recognize the particular preparation. All he can see is a randomly polarized photon.

Bob has the second stage under control, the center part of Fig. 1, where he performs one of the six projections (3). For projection on state $|'\mathsf{h}'\rangle$, the PBS alone suffices, since it reflects v polarized photons and transmits h polarized ones. For all other projections, a suitably set HWP, or a quarter wave plate (QWP), or both are used to turn the polarization in question into h. The then transmitted photon is h polarized, and by passing it through a second HWP, or QWP, or both its polarization is turned back to the wanted one. In this manner, each of the six projections (3) can be implemented by Bob. If his projection is successful, the photon is forwarded to the third stage; otherwise Bob has to ask Alice to prepare another photon and the procedure must be repeated [10].

The third stage, the bottom part of Fig. 1, begins with the conversion of the longitudinal alternative of arriving "early or late" (E or L) into the transversal alternative of moving "downwards or to the right" (D or R). This is achieved with the aid of the beam splitter (BS) and two mirrors to the left of the dashed curve. The E component takes the detour over the mirrors (of the same length as the one in the top part) and becomes D; the L component goes straight ahead and becomes R. In the other cases—E going straight ahead or L taking the detour—the photon will arrive at one of the detectors either before or after the time interval during which they are gated in accordance with the trigger impulse from the first stage. The path length difference of 90 cm translates into a time delay of 3 ns, which can be conveniently resolved by an electronic gate window of 1.2 ns [9].

As soon as the $E \to D$, $L \to R$ conversion is accomplished (and the dashed curve is reached in the bottom part of Fig. 1), the VAA basis of (4) could be measured either with a MZI with nonpolarizing BSs [7], or with PBSs, as was chosen here because of the easier fine-tuning of the 50:50 beam splitting. An additional interferometer loop connects the conversion stage with the VAA analyzer. The whole setup thus consists of two consecutive MZIs, where the first has a BS at the input port and a PBS at the output port. The latter serves also as the input port of the second MZI, which has another PBS at its output port. After emerging at one of the two output channels of the second MZI, the photon passes through yet another PBS and is then detected by one of four detectors. All HWPs are oriented at 22.5° or −22.5° such that a photon that arrives in one of the four VAA states (4) is guided to the corresponding photodetector.

The setup shown in Fig. 1 is schematic. In the real implementation the mirrors of the various MZIs are retroreflecting prisms such that a single BS can serve as input and output components. Additional compensator plates (not indicated in Fig. 1) correct for birefringence of BSs and prisms. And each of the four interferometer loops is phase locked with a He-Ne reference laser.

Once one of the detectors has fired (in the gate window), Alice knows immediately which projection was performed by Bob if he chose between + and −, what it was if he chose between r and l, and what it was if he chose between h and v. The outcome of one run of the experiment is shown in Fig. 2. After the wave plates of the second stage of Fig. 1 are set properly to effect one of the six projections of (3), the counts of the gated detectors of the third stage are recorded for a duration of 30 s. Upon determining the respective fractions of clicks by the wrong detectors, one infers Alice's experimental odds for guessing Bob's projection right. Her success probability exceeds in each case the 90.2% odds of the optimal single-qubit strategy. On average the odds are 95.6%, with a statistical error of ±1.2%.

Imperfections of the optical elements and of their alignment result in occasional clicks by a detector that should not fire and, therefore, the projection is not always inferred correctly, as the data reported in Fig. 2 show. But we do get consistently better odds of guessing the projection right than the 90.2% that the best single-qubit strategy would offer. Indeed, there is a true payoff from entangling the photon polarization with a spatial alternative of the photon (first "E or L"—then "D or R"), and we have succeeded in realizing the mean king's problem by quantum-optical means.

It may be worthwhile to state explicitly how the challenge would be phrased in the particular context of our experimental setup. First Bob would choose one of the six projections, set the wave plates of his stage fittingly, and then tell Alice to send the photon. She follows suit, prepares a photon at the first stage, and waits for one of the detectors of the third stage to fire. It is important that Bob also has access to the knowledge if a photon has been detected or not. If no detector fires, Alice has to send a second photon, and a third (and fourth, fifth, ...) if necessary. But as soon as one photon is detected, Bob calls in Alice's guess for the chosen projection, and this run is over.

Rather than the VAA basis of (4), Alice can just as well measure the other VAA basis [11,12] that consists of the complementary set of states $|-\mathrm{lv}\rangle$, $|-\mathrm{rh}\rangle$, $|+\mathrm{lh}\rangle$, and

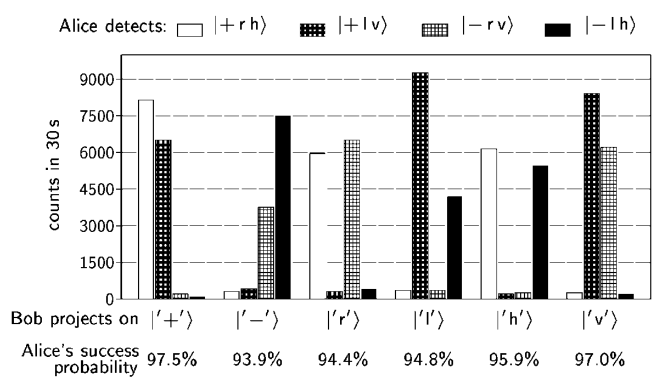

FIG. 2. Outcome of a run of the experiment in which the clicks of the gated detectors of the third stage in Fig. 1 are counted for 30 s for each of the projections (3); see text. Alice's inferred odds for guessing Bob's projection right exceed, in each case, the 90.2% odds of the optimal single-qubit strategy, with average odds of 95.6% ± 1.2%.

$|+\mathrm{rv}\rangle$. In the setup of Fig. 1, one needs only to change by $\frac{1}{2}\pi$ the phase of the polarizing MZI of the VAA analyzer, and by π the phase of the connecting MZI loop, by adjusting some arm lengths. We have performed the projection-retrodiction experiment also with this other VAA basis and have obtained results similar to those reported in Fig. 2, with average odds of 94.7% ± 1.2%.

In the *mean king's second challenge* [11], Bob does not choose between one of six polarization projections, but rather between one of six unitary polarization transformations—three pairs of two:

$$|\mathrm{init}\rangle \to 2^{-1/2}(\sigma_x \pm \sigma_y)|\mathrm{init}\rangle = \begin{cases} (\frac{1}{2}i)^{1/2}(|\mathrm{E,v}\rangle - i|\mathrm{L,h}\rangle), \\ (2i)^{-1/2}(|\mathrm{E,v}\rangle + i|\mathrm{L,h}\rangle), \end{cases}$$
$$|\mathrm{init}\rangle \to 2^{-1/2}(\sigma_y \pm \sigma_z)|\mathrm{init}\rangle = \begin{cases} 2^{-1/2}(|\mathrm{E,r}\rangle - i|\mathrm{L,l}\rangle), \\ -2^{-1/2}(|\mathrm{E,l}\rangle + i|\mathrm{L,r}\rangle), \end{cases}$$
$$|\mathrm{init}\rangle \to 2^{-1/2}(\sigma_z \pm \sigma_x)|\mathrm{init}\rangle = \begin{cases} 2^{-1/2}(|\mathrm{E,+}\rangle + |\mathrm{L,-}\rangle), \\ 2^{-1/2}(|\mathrm{E,-}\rangle - |\mathrm{L,+}\rangle), \end{cases} \quad (5)$$

and Alice has to find out which of the two transformations of a pair was actually performed after eventually being told which of the three pairs applies. In the corresponding experimental setup, only Bob's second stage is different; Alice's preparation in the first stage and her detection in the third stage remain exactly the same. Another difference is the threshold set by the optimal single-qubit strategy. It is only $\frac{5}{6} = 83.3\%$ for the second challenge. Our experimental guessing odds had average success probabilities that were consistently in excess of this threshold, with average odds of 92.2% ± 0.7%.

In summary, then, the 1987 spin-retrodiction puzzle by Vaidman, Aharonov, and Albert—the mean king's problem—has been realized in the form of a quantum-optical analog, in which one infers which polarization projection was performed on a single photon. We have achieved success probabilities that exceed, in each channel, the single-qubit optimum. Further, we have successfully implemented the mean king's second challenge, in which unitary polarization changes are performed rather than polarization projections. The realization of this quantum game is based on implementing single-photon quantum logic [7], and is thus a first step toward more complex tasks of all-photonic quantum computers [13].

As an outlook we note that the mean king's problem suggests deterministic schemes for quantum cryptography [12] and for direct secure quantum communication "with a publicly known key" [11,14]. A demonstration experiment for the cryptography scheme is already feasible with an apparatus that differs only by the simultaneously implemented second VAA basis analyzer.

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[4] Alice prepares the photon in a certain polarization state specified by the statistical operator $\rho = \frac{1}{2}(1 + x\sigma_x + y\sigma_y + z\sigma_z)$ with $x^2 + y^2 + z^2 \leq 1$. In her control measurement, she measures $\vec{e} \cdot \vec{\sigma}$ with some unit vector $\vec{e} = (e_x, e_y, e_z)$. After being told which intermediate measurement was performed by Bob, she bets on the outcome that contributes most to the probability that the result actually found appears in her control measurement. Summed over all possible scenarios, her total odds for guessing right are

$$\frac{1}{2} + \frac{1}{6}\sum_{\zeta=x,y,z} \max\{|\zeta|, |e_\zeta|\} \leq \frac{2}{3} + \frac{1}{6}\sqrt{2} = 90.2\%.$$

The maximum of 90.2% is achieved for $(x, y, z) = (0, 1, 0)$ and $\vec{e} = (2^{-1/2}, 0, 2^{-1/2})$, for example, which corresponds to the situation described in the text.

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[8] In the proposal of [7] the von Neumann measurement is mimicked by following the destructive photodetection with a suitable unitary polarization change on another photon. An experiment along these lines would require very fast switches and, therefore, we modified the scheme and replaced measurements by projections. See also [9].

[9] Only 50% of the incoming photons are detected during the gate time. This can be avoided by replacing the first BS of the third stage by a polarization insensitive, quickly switchable mirror. While such a device is conceptually possible, one would lose efficiency and signal-to-noise ratio with present-day technology.

[10] Bob's projection is successful in 50% of the cases. With perfect detectors, he would note that his preparation failed upon registering the photon reflected from his PBS.

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